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Non-Time Additive Utility Optimization — the Case of Certainty

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Abstract

We study the intertemporal utility maximization problem for Hindy-Huang-Kreps utilities. Necessary and sufficient conditions for optimality are given. An explicit solution is provided for a large class of utility functions. In particular, the case of separable power utilities with a finite time horizon is solved explicitly.

Keywords: non-time additive utility optimization, intertemporal utility

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1 Introduction

In an essay to overcome the deficiencies of the received theory, Hindy, Huang and Kreps [2] (henceforth, HHK) presented a fundamentally new approach to intertemporal utility theory. They replaced the standard consumption space with the space of right-continuous, increasing functions and proposed to use a new class of utility functionals since the standard time-additive utility functionals are not continuous in the economically appropriate topology. Here, we present a new method for solving the utility maximization problem associated with the class of utility functionals HHK introduced. Our aim is to give necessary and sufficient first order conditions for optimality. Moreover, we will characterize explicitly the solution for a large class of utilities. The special case of separable power utilities within a finite time horizon is solved in closed form.

When consumption problems in continuous time are considered, agents are typically assumed to consume all in a final gulp or to consume continuously at rates. In the latter case, the utility functional is usually assumed to be of some time-additive form. This approach has several difficulties. First, there is no reason, a priori, to exclude the possibility of consumption both in gulps and in rates. It is therefore more natural to allow for jumps in cumulated consumption plans. Hence, the natural consumption space is the space of all increasing, right-continuous processes. Second, in the classical approach, resulting equilibrium prices are in general not continuous in time, whereas it seems plausible to consider consumption at nearby dates as substitutes which should, therefore, have similar prices. Moreover, the existence of an interest rate is not ensured by the time-additive model. Technically, this is due to the fact that standard time-additive utility functionals are not continuous in the weak topology, as HHK showed.

Unsatisfied with the time-additive approach, HHK develop a new framework for intertemporal utility theory. They consider, as mentioned above, a larger consumption space, namely the space of all distribution functions which are interpreted as cumulative consumption plans. The authors suggest to use a new type of utility functional in which the argument of the felicity function is some index of past consumption rather than of current consumption alone. HHK give sufficent conditions for optimality in the corresponding utility maximization problem by using the Bellman methodology. In particular, they solve explicitly for the case of power utilities within the framework of constant interest rates and an infinite time horizon.

We propose a new, more static, approach to solve the corresponding optimization problem which avoids the Bellman methodology. Instead, we provide an analog of the Kuhn-Tucker theorem for this infinite-dimensional problem which yields necessary and sufficient first-order conditions. Endowed with this theorem, we explicitly solve the utility maximization problem for the case of a finite time horizon.

The paper is organized as follows. In the next section, we formulate the problem to be solved and show that it is equivalent to the dynamic problem studied by [2]. Section 3 demonstrates one of the advantages of the HHK approach: the utility functional is continuous and budget sets are compact in the appropriate topology. Hence, existence and uniqueness of solutions are easily shown. In Section 4, we give Kuhn-Tucker-like necessary and sufficient conditions for optimality. Section 5 develops a method for solving such utility maximization problems and characterizes completely the optimal solution for a large class of utilities. Section 6 illustrates the procedure by appplying it to the case of power utilities in the finite horizon case. This case is led to a complete closed-form solution.

2 Formulation of the Problem

An economic agent living from time 0 up to some time $T \geq 0$ decides how much of a perishable consumption good to consume at each time $t \in [0, T]$. Following HHK, we denote the set of all possible consumption plans by

$$\mathcal{C} \stackrel{\Delta}{=} \{C : [0, T] \to \mathbb{R} \text{ nonnegative, nondecreasing and rightcontinuous}\}.$$

Instead of studying the dynamic problem set up by HHK, we consider a static problem. As in the Arrow-Debreu framework, we assume as given a complete set of forward markets, where the consumption good is traded at some deterministic price $\psi(t)$ $(0 \leq t \leq T)$. The agent buys his preferred consumption plan at time 0.

We assume that ψ is a continuous, strictly positive function. Then, the corresponding price functional[1] $\Psi(C) \stackrel{\Delta}{=} \int_0^T \psi(u)dC(u)$ is a linear functional on $\mathcal{C}$ which is continuous in the weak topology. The agent is endowed with some capital $w \geq 0$ and his budget set is given by

$$\mathcal{A}(w) \stackrel{\Delta}{=} \{C \in \mathcal{C} \text{ such that } \Psi(C) \leq w\}.$$

Remark 2.1 *This formulation is equivalent to the dynamic one considered by HHK. In their setup, $\psi(t) = e^{-rt}$, and the agent's wealth $w(t)$ $(0 \leq t \leq T)$ is defined by the intertemporal budget constraint $dw(t) = rw(t) - dC(t)$ and has to be nonnegative (no ruin). It follows that $w(t) = e^{rt}\left(w(0-) - \int_0^t e^{-rs}dC(s)\right)$. The no ruin condition implies $0 \leq w(T) \leq e^{rT}\left(w(0-) - \int_0^T e^{-rs}dC(s)\right)$, hence $\Psi(C) \leq w(0-) = w$.*

In contrast to the standard models, the agent does not obtain utility from his current consumption $dC(t)$, but from an index of past consumption Y^C. Following HHK we assume this index to be given by

$$Y^C(t) \stackrel{\Delta}{=} y(t) + \int_0^t \theta(t,s)dC_s$$

[1]Here, and in the rest of the paper, the range of the integral is the whole closed interval $[0, T]$, unless otherwise indicated.

for some nonnegative, continuous functions θ and y. A typical example, which we will study later on, is the familiar habit formation index $\theta(t,s) = \beta e^{-\beta(t-s)}$, confer [1] and [3]. y represents some exogenously given standard of living.

The utility associated with a given consumption plan $C \in \mathcal{C}$ is given by the functional

$$U(C) \triangleq \int_0^T u(t, Y^C(t))\, dt \tag{1}$$

where $u(.,.)$ denotes a continuous function $[0,T] \times \mathbb{R}^+ \to \mathbb{R}$.

The remainder of the paper studies the agent's utility maximization problem

$$\text{Maximize } U(C) \text{ over C subject to } C \in \mathcal{A}(w). \tag{2}$$

3 Existence and Uniqueness of an Optimal Consumption Plan

In a first step, we approach the utility maximization problem from a general perspective and establish existence and uniqueness of optimal consumption plans. This is done easily in the present setup, since, as in the finite dimensional case, budget sets are compact and utility functionals are continuous with respect to the weak topology. It may be interesting to note the problems which arise with classical utility functions. These are, if at all, continuous only with respect to the variational norm topology. But budget sets are not compact in this strong topology. Therefore, additional assumptions are needed to establish existence of optimal solutions. In this sense, HHK provide an elegant and convenient framework for intertemporal utility maximization.

Proposition 3.1 *The utility functional $U(.)$ is continuous on $\mathcal{C}$ equipped with the topology of weak convergence of measures on $([0,T], \mathcal{B}([0,T]))$.*

PROOF: Since the topology of weak convergence of measures on $([0,T], \mathcal{B}([0,T]))$ is metrizable (e.g., by the Prohorov metric) it suffices to consider a weakly convergent sequence of measures $C^n \in \mathcal{C}$ $(n = 0, 1, \ldots)$ and show that the corresponding utilities $U(C^n)$ converge to the utility $U(C)$ of the limiting measure $C \triangleq \lim_n C^n$.

For every fixed $t \in [0,T]$ such that $\Delta C(t) = 0$ the function $s \mapsto \theta(t,s) 1_{[0,t]}(s)$ is continuous in dC-a.e. $s \in [0,T]$. Therefore, by the Portmanteau Theorem, we have $Y^{C^n}(t) \to Y^C(t)$ for each such t, i.e. in particular for Lebesgue a.e. $t \in [0,T]$. By weak convergence we have $C^n(T) \to C(T)$ implying the uniform boundedness of $C^n(t)$ and consequently the uniform boundedness of $Y^{C^n}(t)$ $(0 \le t \le T, n = 0, 1, \ldots)$. Now the assertion follows by use of dominated convergence. □

Proposition 3.2 *$\mathcal{A}(w)$ is compact with respect to the weak topology.*

PROOF: It suffices to consider a sequence $C^n \in \mathcal{A}(w) - \{0\}$ $(n = 0, 1, \ldots)$ and show that there is a convergent subsequence. For this we note that $P^n \triangleq C^n / C^n(T)$ defines a sequence of probability measures on the compact interval $[0, T]$. Thus, by Prohorov's theorem, there is a subsequence along which the sequence P^n $(n = 0, 1, \ldots)$ converges weakly to some probability measure P on $[0, T]$. Since $C^n \in \mathcal{A}(w)$ implies $0 \leq C^n(T) \leq w / \min \psi$ we can extract a further subsequence, again denoted by n, such that also $C^n(T)$ converges. It follows the weak convergence of $C^n = C^n(T) P^n$ along this subsequence. □

Theorem 3.3 *There is a solution $C^* \in \mathcal{A}(w)$ to the optimization problem* (2)*. If $u(t, .)$ is strictly monotone and strictly concave for every $t \in [0, T]$ and if $C \mapsto Y^C$ is injective this solution is unique.*

PROOF:

(i) The first statement follows immediately from Propositions 3.1 and 3.2.

(ii) Let $\tilde{C} \neq C^*$ be another optimal consumption plan. Then by rightcontinuity (and optimality[2]) there must be an open interval on which $Y^{\tilde{C}} \neq Y^{C^*}$.

Now, if $u(t, .)$ is strictly concave for all $t \in [0, T]$ this implies for $\lambda \in (0, 1)$

$$\begin{aligned} U(\lambda \tilde{C} + (1 - \lambda) C^*) &= \int_0^T u(t, \lambda Y^{\tilde{C}}(t) + (1 - \lambda) Y^{C^*}(t))\, dt \\ &> \int_0^T \{\lambda u(t, Y^{\tilde{C}}(t)) + (1 - \lambda) u(t, Y^{C^*}(t))\}\, dt \\ &= \lambda U(\tilde{C}) + (1 - \lambda) U(C^*) \\ &= \max_{C \in \mathcal{A}(w)} U(C) \end{aligned}$$

contradicting the optimality of C^* over $\mathcal{A}(w)$.

□

Remark 3.4 *For $C \mapsto Y^C$ to be injective it suffices, e.g., to assume that $\theta(., .)$ is strictly positive and separable, i.e. $\theta(t, s) = \eta(t) \kappa(s)$ $(0 \leq s \leq t \leq T)$ for some strictly positive, continuous functions η, κ.*

4 Sufficient and Necessary Conditions for Optimal Consumption Plans

We have already seen that the preferences associated with (1) are continuous. To ensure that they satisfy also the standard properties of convexity and local nonsatiation, we make the following standing

[2]The case $Y^{\tilde{C}} \neq Y^{C^*}$ *only* for $t = T$ is excluded since it implies that $\tilde{C}$ or C^* must have a real jump in T — a property obviously contradicting optimality, if $u(t, .)$ is strictly monotone.

Assumption 4.1 *For all $t \in [0, T]$ the felicity function $u(t, .)$ is strictly monotone and concave. The derivative $\partial_y u(t, y)$ exists and $(t, y) \mapsto \partial_y u(t, y)$ is continuous on* $[0, T] \times (0, \infty)$.

We are now in the position to formulate the announced Kuhn-Tucker-like necessary and sufficient first-order conditions for optimality.

Theorem 4.2 *Necessary and sufficient conditions for a consumption plan $C^* \in \mathcal{A}(w)$ to solve the optimization problem* (2) *are*

$$
(3) \quad \begin{cases} \text{(i)}\ \Psi(C^*) = w \\ \text{(ii)}\ \int_t^T \partial_y u(s, Y^{C^*}(s))\theta(s,t)\, ds \le M\psi(t) \quad \forall t \in [0,T] \\ \text{(iii)}\ \int_t^T \partial_y u(s, Y^{C^*}(s))\theta(s,t)\, ds = M\psi(t) \quad \forall t \in \text{supp } dC^* \end{cases}
$$

for some constant $M = M(w, T) \ge 0$.

Proof: Consider $C \in \mathcal{A}(w)$ and let $Y \triangleq Y^C$, $Y^* \triangleq Y^{C^*}$. By concavity of u and definition of Y and Y^*, one has

$$
\begin{aligned}
U(C^*) - U(C) &= \int_0^T \{u(s, Y^*(s)) - u(s, Y(s))\}\, ds \\
&\ge \int_0^T \{\partial_y u(s, Y^*(s))(Y^*(s) - Y(s))\}\, ds \\
&= \int_0^T \{\partial_y u(s, Y^*(s)) \int_0^s \theta(s,t)\,[dC^*(t) - dC(t)]\}\, ds\,.
\end{aligned}
$$

Fubini's theorem and the first-order conditions (3) (ii) and (iii) allow to conclude

$$
\begin{aligned}
U(C^*) - U(C) &\ge \int_0^T \{\int_t^T \partial_y u(s, Y^*(s))\theta(s,t)\, ds\}\,[dC^*(t) - dC(t)] \\
&\ge M \int_0^T \psi(t)\,[dC^*(t) - dC(t)] \\
&\ge 0
\end{aligned}
$$

where the last inequality is due to the budget constraint and (3) (i). This proves sufficiency.

The necessity part is more difficult and is provided by the following lemmata. The idea is that an optimal plan C^* solves also the problem linearized around C^* (Lemma 4.3). The solutions of the linear problem are easily characterized (Lemma 4.4), and necessity follows (Corollary 4.5). □

Lemma 4.3 *Let $C^* \in \mathcal{A}(w)$ be optimal for* (2) *and let*

$$
(4) \qquad f^*(t) \triangleq \int_t^T \partial_y u(s, Y^{C^*}(s))\theta(s,t)\, ds \quad (0 \le t \le T).
$$

Then C^* solves the linear problem

$$\text{Maximize } \int_0^T f^*(t)\, dC(t) \text{ over } C \text{ subject to } C \in \mathcal{A}(w).$$

PROOF: Consider $C \in \mathcal{A}(w)$ and let $C^\epsilon \triangleq \epsilon C + (1-\epsilon)C^*$. By optimality of C^* and concavity of $u(t,.)$ for all $t \in [0,T]$ we have for $Y^\epsilon \triangleq Y^{C^\epsilon}$, $Y \triangleq Y^C$ and $Y^* \triangleq Y^{C^*}$

$$\begin{aligned}
0 &\geq \frac{1}{\epsilon}\{U(C^\epsilon) - U(C^*)\} \\
&= \int_0^T \frac{1}{\epsilon}\{u(s, Y^*(s) + \epsilon(Y(s) - Y^*(s))) - u(s, Y^*(s))\} ds \\
&\geq \int_0^T \{\partial_y u(s, Y^\epsilon(s))(Y(s) - Y^*(s))\}\, ds \\
&= \int_0^T \{\partial_y u(s, Y^\epsilon(s)) \int_0^s \theta(s,t)\,[dC(t) - dC^*(t)]\}\, ds \\
&= \int_0^T \{\int_t^T \partial_y u(s, Y^\epsilon(s))\theta(s,t)\, ds\}\,[dC(t) - dC^*(t)] \\
&\xrightarrow{\epsilon \downarrow 0} \int_0^T \{\int_t^T \partial_y u(s, Y^*(s))\theta(s,t)\, ds\}\,[dC(t) - dC^*(t)]
\end{aligned}$$

by dominated convergence. Rearranging terms we get

$$\int_0^T f^*(s)\, dC^*(s) \geq \int_0^T f^*(s)\, dC(s).$$

□

Lemma 4.4 *Let $f, g \in C^0([0,T])$ with $g > 0$ on $[0,T]$. Then the solutions to the linear optimization problem*

$$\text{Maximize } \int_0^T f(s)\, dC(s) \text{ over } C \text{ subject to } \int_0^T g(s)\, dC(s) \leq 1.$$

are precisely those $\hat{C} \in \mathcal{C}$ such that

(i) $\int_0^T g(t)\, d\hat{C}(t) = 1$,

(ii) $\operatorname{supp} d\hat{C} \subset \arg\max \frac{f}{g}$.

PROOF: Let $M \triangleq \max \frac{f}{g}$ and consider $C \in \mathcal{C}$ such that $\int_0^T g(s)\, dC(s) \leq 1$. We have

$$\int_0^T f(t)\, dC(t) = \int_0^T \frac{f(t)}{g(t)} g(t)\, dC(t) \leq \int_0^T M g(s)\, dC(s) \leq M$$

with equality if and only if C satisfies (i) and (ii). □

Corollary 4.5 *Every solution $C^* \in \mathcal{A}(w)$ of* (2) *satisfies*

$$\begin{cases} \text{(i)} \int_0^T \psi(t)\, dC^*(t) = w \\ \text{(ii)} \ \operatorname{supp} dC^* \subset \arg\max \phi^{C^*} \end{cases}$$

where for $C \in \mathcal{C}$ we set

(5) $$\phi^C(t) \triangleq \frac{1}{\psi(t)} \int_t^T \partial_y u(s, Y^C(s))\theta(s,t)\, ds \quad (0 \le t \le T).$$

PROOF: This follows with $f \triangleq f^*$ as in (4) and $g(t) \triangleq \psi(t)/w$ $(0 \le t \le T)$ by the above lemmata. □

5 Explicit Solution for a Large Class of Utilities

In this section we present a general method for solving the optimization problem (2). This method works if the price density ψ is given by

$$\psi(t) = e^{-rt} \ (0 \le t \le T)$$

for some constant $r \ge 0$ and if the investor's preferences are as follows.

Assumption 5.1 *(i) For $C \in \mathcal{C}$ the investor's index of past consumption Y^C is given by*

(6) $$Y^C(t) \triangleq y e^{-\beta t} + \beta \int_0^t e^{-\beta(t-s)}\, dC(s) \quad (0 \le t \le T)$$

for some constants $y, \beta > 0$.

(ii) The investor's felicity function u and its first and second partial derivatives are continuous on $[0,T] \times (0,+\infty)$. Furthermore $u = u(t,y)$ is strictly concave in $y > 0$ and

$$\partial_y u(t, 0+) = +\infty, \quad \partial_y u(t, +\infty) = 0 \quad (0 \le t \le T).$$

Finally we have

$$\mathcal{L}u(t,y) > 0 \quad (0 \le t \le T, y > 0)$$

where $\mathcal{L}$ is the differential operator defined by

$$\mathcal{L} \triangleq r\partial_y - \beta y \partial_y^2 + \partial_t \partial_y.$$

Remark 5.2 *(i) The choice of*

$$\theta(t,s) \stackrel{\Delta}{=} \beta e^{-\beta(t-s)} \quad \text{and} \quad y(t) \stackrel{\Delta}{=} y e^{-\beta t} \qquad (0 \le s \le t \le T)$$

links the above setting to the preceding sections. As pointed out in Section 2 it gives us a familiar habit formation index which has been studied in the context of a notably different utility functional by, e.g., [1], [3]. In our context it has been proposed and treated by HHK. This index evolves according to the differential equation

$$Y^C(0-) = y, \quad dY^C(t) = \beta(dC(t) - Y^C(t)\,dt) \quad (0 \le t \le T)$$

which allows an easy interpretation and a comfortable analysis.

(ii) By Assumption 5.1 $\partial_y u(t,.)$ *is a strictly decreasing, continuous and surjective function* $(0,\infty) \to (0,\infty)$ *for every* $t \in [0,T]$. *Therefore it allows an inverse which we denote by* $i(t,.) : (0,\infty) \to (0,\infty)$. *Of course,* $i(.,.)$ *has the same degree of smoothness as* $\partial_y u(.,.)$.

(iii) In terms of rate of time preference $\delta \stackrel{\Delta}{=} -\frac{\partial_t \partial_y u(t,y)}{\partial_y u(t,y)}$, *interest rate* r, *relative risk aversion* $a \stackrel{\Delta}{=} -\frac{y\partial_y^2 u(t,y)}{\partial_y u(t,y)}$ *and rate of decay* β, *the condition* $\mathcal{L}u(t,y) > 0$ *is equivalent to* $\delta < r + \beta a$. *Hence, we assume that the investor's rate of time preference is sufficiently small as compared to interest rate, risk aversion and rate of decay. As we will see later on in an example (confer Remark 6.1), a high rate of time preference induces the consumer to take a large gulp and to stop consuming afterwards.*

Obviously, our optimality criterion (3) does not give an explicit description of a solution to (2). The main problem seems to be that one does not know enough about the structure of the support of an optimal consumption plan C^*. In fact, one only has that this set has to be contained in $\arg\max \phi^*$ where

$$\phi^* \stackrel{\Delta}{=} \phi^{C^*} = e^{(r+\beta)t} \int_t^T \partial_y u(s, Y^{C^*}(s))\beta e^{-\beta s}\,ds \quad (0 \le t \le t)$$

as defined in Corollary 4.5. The following theorem shows that under Assumption 5.1 these two sets coincide and that they are a closed interval in $[0,T]$. Thus ϕ^* may be viewed as a "consumption signal" which characterizes the time periods when the investor should consume.

Theorem 5.3 *Under Assumption 5.1 we have* $\operatorname{supp} dC^* = \arg\max \phi^*$ *which is a closed interval in* $[0,T]$.

PROOF:

1. We first show that supp dC^* is a closed interval. Otherwise there are $t_0, t_1 \in$ supp dC^* with $t_0 < t_1$ satisfying $(t_0, t_1) \cap \operatorname{supp} dC^* = \emptyset$. This implies that ϕ^* is smooth on (t_0, t_1) with first and second derivatives given by

$$\partial_t \phi^*(t) = (r+\beta)\phi^*(t) - e^{rt}\partial_y u(t, Y^*(t))$$

and

$$\partial_t^2 \phi^*(t) = (r+\beta)\partial_t \phi^*(t) - e^{rt}\mathcal{L}u(t, Y^*(t)) \tag{7}$$

respectively. By Corollary 4.5 we know that t_0 and t_1 maximize ϕ^*. Hence the minimum of ϕ^* over $[t_0, t_1]$ must be attained at some interior point $\hat{t} \in (t_0, t_1)$. Since $\hat{t}$ locally minimizes the smooth function ϕ^* we must have

$$\partial_t \phi^*(\hat{t}) = 0, \quad \partial_t^2 \phi^*(\hat{t}) \geq 0.$$

But using $\partial_t \phi^*(\hat{t}) = 0$ and (7) we get that

$$\partial_t^2 \phi^*(\hat{t}) = -e^{r\hat{t}}\mathcal{L}u(\hat{t}, Y^*(\hat{t})) < 0$$

by Assumption 5.1 (ii) — a contradiction.

2. It remains to show that supp $dC^* = \arg\max \phi^*$. By 1. we know that there are $\underline{t}, \overline{t} \in [0, T]$ such that supp $dC^* = [\underline{t}, \overline{t}]$. By Corollary 4.5 we know that supp $dC^* \subset \arg\max \phi^*$. Therefore, if the above assertion is wrong, there must be some $\hat{t} \in \arg\max \phi^*$ such that either $\hat{t} < \underline{t}$ and $[\hat{t}, \underline{t}) \cap \operatorname{supp} dC^* = \emptyset$ or $\overline{t} < \hat{t}$ and $(\overline{t}, \hat{t}] \cap \operatorname{supp} dC^* = \emptyset$. Replacing (t_0, t_1) by $(\hat{t}, \underline{t})$ ($(\overline{t}, \hat{t})$ respectively) one obtains a contradiction to Assumption 5.1 (ii) using the same argument as in 1.

$\square$

In connection with our optimality criterion (3) the above theorem implies the following structure for the optimal consumption index Y^{C^*} our investor can achieve. There are $\underline{t}, \overline{t} \in [0, T]$ with $\underline{t} \leq \overline{t}$ such that

- the investor does not consume before $\underline{t}$ and after $\overline{t}$, hence Y^{C^*} decays exponentially at rate $-\beta$ on $[0, \underline{t}) \cup (\overline{t}, T]$
- for $t \in [\underline{t}, \overline{t}]$ we have

$$\beta e^{(r+\beta)t} \int_t^T \partial_y u(s, Y^{C^*}(s)) e^{-\beta s}\, ds = M,$$

which yields upon differentiating

$$Y^{C^*}(t) = I^M(t) \triangleq i\left(t, M\frac{r+\beta}{\beta}e^{-rt}\right) \tag{8}$$

for some constant $M > 0$.

This suggests the following strategy for finding an optimal consumption plan. For each $M > 0$ determine $\underline{t}(M), \overline{t}(M) \in [0,T]$ such that the process of bounded variation C^M with supp $dC^M = [\underline{t}(M), \overline{t}(M)]$ and $Y^{C^M}(t) = I^M(t)$ on $[\underline{t}(M), \overline{t}(M)]$ is indeed a nondecreasing, nonnegative process with $\arg\max \phi^{C^M} = \operatorname{supp} dC^M$. Choose then $M^* > 0$ such that C^{M^*} meets the investor's budget constraint precisely, i.e. $\Psi(C^{M^*}) = w$.

Remark 5.4 *An easy calculation shows that on open intervals where Y^{C^*} is given by* (8) *we must have*

$$\beta\, dC^*(t) = dY^{C^*}(t) + \beta Y^{C^*}(t)\, dt = -\frac{\mathcal{L}u}{\partial_y^2 u}(t, I^M(t))\, dt$$

which is strictly positive by Assumption 5.1. Similarly, one can see

$$\frac{\partial}{\partial t}(I^M(t)e^{\beta t}) = -e^{\beta t}\frac{\mathcal{L}u}{\partial_y^2 u}(t, I^M(t)) > 0 \quad (t \in [0,T]), \tag{9}$$

a property which will prove useful later. This underlines once more the role condition $\mathcal{L}u > 0$ plays, since it ensures the process C^M associated to Y^{C^M} to be nondecreasing on intervals where $Y^{C^M} = I^M$.

In order to construct the above $\underline{t}(M)$ and $\overline{t}(M)$ let us define the auxiliary functions

$$\phi^{M,t}(s) \triangleq e^{(r+\beta)s}\int_s^T \partial_y u(v, I^M(t)e^{-\beta(v-t)})\beta e^{-\beta v}\, dv \quad (t \le s \le T)$$

and $\Phi^M(t) \triangleq \phi^{M,t}(t)$ where $M > 0$ and $t \in [0,T]$. $\phi^{M,t}(.)$ describes how the consumption signal evolves if the investor refrains from consumption from time t on when his consumption index equals $I^M(t)$, c.f. (8). The optimality criterion (3) and Theorem 5.3 suggest that the last time of consumption is given by the minimal t such that $\phi^{M,t}(s)$ attains its maximum over $[t,T]$ at time $s = t$. Since this description is far from being explicit we would like to give a different characterization of $\overline{t}(M)$. This is done in

Lemma 5.5 *Each $M > 0$ has one and only one of the following two properties: either*

(i) There is $t^* \in (0,T]$ such that $\Phi^M(t^*) = M$.

or

(ii) $\partial_s \phi^{M,t}(s) \le 0$ for all $t \in [0,T]$, $s \in [t,T]$.

In addition, if (i) holds the corresponding t^* is uniquely determined and we have $\partial_s \phi^{M,t^*}(s) \le 0$ for all $s \in [t^*,T]$.

PROOF:

1. For $0 \le t \le s \le T$ we have

$$\partial_s \phi^{M,t}(s) = (r+\beta)\phi^{M,t}(s) - \beta e^{rs} \partial_y u(s, I^M(t) e^{-\beta(s-t)}).$$

Hence, using the monotonicity of $\partial_y u(s,.)$ and of $I^M(t)e^{\beta t}$ $(0 \le t \le T)$ (confer Remark 5.4) we get

$$\partial_s \phi^{M,t}(s) \le (r+\beta)[\phi^{M,t}(s) - M] \quad \forall\ 0 \le t \le s \le T \tag{10}$$

by definition of $i(s,.) = \partial_y u(s,.)^{-1}$.

2. Suppose that (i) does not hold. Then we must have $\Phi^M(t) = \phi^{M,t}(t) \le M$ for all $t \in [0,T]$ because $\Phi^M(.)$ is a continuous function with $\Phi^M(T) = 0 < M$. Therefore proceeding from (10) we see:

$$\partial_s \phi^{M,t}(s) \le (r+\beta)[\phi^{M,t}(s) - \phi^{M,t}(t)] \quad \forall\ 0 \le t \le s \le T$$

or, equivalently,

$$\partial_s \phi^{M,t}(s) \le (r+\beta)\int_t^s \partial_s \phi^{M,t}(v)\, dv \quad \forall\ 0 \le t \le s \le T.$$

Gronwall's Inequality yields (ii).

3. Now let us assume that (i) holds, i.e. there is $t^* \in (0,T]$ with $\Phi^M(t^*) = M$. For $t \in [0,T)$ we have

$$\begin{aligned}
&\partial_t \Phi^M(t) \\
&= \frac{\partial}{\partial s}\phi^{M,t}(s)\Big|_{s=t} + \frac{\partial}{\partial t}\phi^{M,t}(s)\Big|_{s=t} \\
&= (r+\beta)[\phi^{M,t}(t) - M] + e^{(r+\beta)t}\int_t^T \partial_y^2 u(\dots)\frac{\partial}{\partial t}(I^M(t)e^{\beta t})\beta e^{-2\beta v}\, dv \\
&< (r+\beta)[\Phi^M(t) - M]
\end{aligned}$$

where the last inequality is due to Remark 5.4. Therefore, we get $\partial_t \Phi^M(t^*) < 0$. Thus there must be some $\hat{t} \in (0, t^*)$ satisfying $\Phi^M(\hat{t}) > \Phi^M(t^*) = M$. We have

$$\frac{\partial}{\partial s}\phi^{M,\hat{t}}(s)\Big|_{s=\hat{t}} = (r+\beta)[\Phi^M(\hat{t}) - M] > 0,$$

i.e. (ii) can not hold.

4. The above estimate for $\partial_t \Phi^M(.)$ implies that $\Phi^M(.)$ is strictly decreasing at each point $t^* \in (0,T]$ such that $\Phi^M(t^*) = M$. Therefore if there was another point $\hat{t}$ solving this equation there would have to be a third solution inbetween where $\partial_t \Phi^M(.)$ is nonnegative — a contradiction. This proves the asserted uniqueness of t^*. $\partial_s \phi^{M,t^*}(s) \le 0$ for $s \in [t^*, T]$ follows as in 2. with $t = t^*$.

□

The above lemma tells us that

$$(11)\ \bar{t}(M) \triangleq \begin{cases} \text{the unique solution } t^* \text{ of } \Phi^M(t) = M \text{ in } (0,T] \text{ if there is some,} \\ 0 \text{ otherwise} \end{cases}$$

is a well defined function $\bar{t} : (0,\infty) \to [0,T]$. In fact, this function can even be proved to be continuous:

Lemma 5.6 $\bar{t} : (0,\infty) \to [0,T]$ *defined by* (11) *is continuous.*

PROOF:

1. If $M_0 > 0$ is such that $\Phi^{M_0}(t^*) = M_0$ for some $t^* > 0$ we know from the proof of Lemma 5.5 that $\partial_t \Phi^{M_0}(t^*) < 0$. Thus the Implicit Function Theorem yields that the equality $\Phi^M(t) = M$ has a solution also for M in some open neighborhood of M_0 that depends continuously on M. This proves the continuity of $\bar{t}(.)$ at such a point M_0.

2. Let us now prove continuity also in points $M_0 > 0$ where there is no solution to $\Phi^{M_0}(t) = M$ in $(0,T]$. For this it suffices to consider a sequence of values $M_n \to M_0 > 0$ $(n \uparrow \infty)$ with corresponding $t_n \in (0,T]$ such that $\Phi^{M_n}(t_n) = M_n$ $(n = 0,1,\dots)$ and prove that we must have $t_n \to 0$ $(n \uparrow \infty)$. Suppose to the contrary that this sequence has an accumulation point $t_0 \in (0,T]$. Since $\Phi^{\cdot}(.)$ is continuous we must have

$$\Phi^{M_0}(t_0) = \lim_n \Phi^{M_n}(t_n) = \lim_n M_n = M_0$$

at least along some suitable subsequence. Thus t_0 is a solution of $\Phi^{M_0}(t) = M_0$ in $(0,T]$ in contrast to our assumption on M_0.

□

The natural time to start consumption is when the marginal utility of consuming is greater than the price of consumption:

$$\begin{aligned} \underline{t}(M) &\triangleq \inf\{t \in [0,T] : \partial_y u(t, ye^{-\beta t}) \geq M \frac{r+\beta}{\beta} e^{-rt}\} \wedge \bar{t}(M) \\ &= \inf\{t \in [0,T] : ye^{-\beta t} \leq I^M(t)\} \wedge \bar{t}(M) \quad (M > 0) \end{aligned} \tag{12}$$

with the convention that $\inf \emptyset \triangleq +\infty$. We have

Lemma 5.7 $\underline{t}(.)$ *is continuous and satisfies* $\underline{t}(M) = 0$ *iff* $0 < M \leq \frac{\beta}{r+\beta} \partial_y u(0,y)$ *and* $\underline{t}(M) = \bar{t}(M)$ *for* $M \geq \frac{\beta}{r+\beta} e^{rT} \partial_y u(T, ye^{-\beta T})$.

PROOF: This is straightforward since $I^M(t)e^{\beta t}$ is continuous in (t,M), strictly decreasing in $M > 0$ and strictly increasing in $t \in [0,T]$. □

Now we can give the main result of this section.

Theorem 5.8 *For each $M > 0$ the uniquely determined process of bounded variation C^M with $C^M(0-) \triangleq 0$, $\operatorname{supp} dC^M \triangleq [\underline{t}(M), \overline{t}(M)]$*[3] *and corresponding*

$$Y^{C^M}(t) \triangleq \begin{cases} I^M(t) \\ y e^{-\beta t} \vee I^M(t) \end{cases} \text{if } \begin{matrix} \underline{t}(M) < \overline{t}(M) \\ \underline{t}(M) = \overline{t}(M) \end{matrix} \quad \text{for } t \in [\underline{t}(M), \overline{t}(M)]$$

is nondecreasing and optimal in its class $\mathcal{A}(\Psi(C^M))$. Furthermore $M \mapsto \Psi(C^M)$ is a continuous, surjective mapping $(0, \infty) \to \mathbb{R}^+$.

PROOF:

1. Let us first show that C^M defined as above is indeed a nondecreasing process. If $\underline{t}(M) = \overline{t}(M) = 0$ we have

$$\Delta C^M(0) = \beta^{-1}(Y^{C^M}(0) - y) = \beta^{-1}(y \vee I^M(0) - y) \geq 0,$$

i.e. C^M is nondecreasing in this case. If $\underline{t}(M) = \overline{t}(M) > 0$ we have $C^M = 0$ which trivially is nondecreasing. It remains the case $\underline{t}(M) < \overline{t}(M)$. To prove that $C^M(t)$ is nondecreasing in $t \in (\underline{t}(M), \overline{t}(M)]$ we refer to Remark 5.4. For $t = \underline{t}(M)$ we only have to check that $\Delta C^M(\underline{t}(M)) \geq 0$. This is clear by definition of $\underline{t}(M)$.

2. Next we prove that C^M is optimal in $\mathcal{A}(\Psi(C^M))$. This is trivial if $\underline{t}(M) = \overline{t}(M) > 0$ or $\underline{t}(M) = \overline{t}(M) = 0$ and $y \geq I^M(0)$ since in both cases we have $C^M = 0$. If $\underline{t}(M) = \overline{t}(M) = 0$ and $y < I^M(0)$ our optimality criterion (3) tells us that it is enough to show $0 \in \arg\max \phi^{C^M}$. Note that for $\overline{t}(M) = 0$

$$\phi^{C^M}(s) = e^{(r+\beta)s} \int_s^T \partial_y u(v, I^M(0)e^{-\beta v})\beta e^{-\beta v}\, dv = \phi^{M,0}(s) \quad (0 \leq s \leq T)$$

which is a nonincreasing function by Lemma 5.5. Let us now consider the case $\underline{t}(M) < \overline{t}(M)$ in which we know that the consumption signal $\phi^{C^M}(s)$ is given for $s \geq \overline{t}(M)$ by

$$\phi^{C^M}(s) = e^{(r+\beta)s} \int_s^T \partial_y u(v, I^M(\overline{t}(M))e^{-\beta(v-\overline{t}(M))})\beta e^{-\beta v}\, dv = \phi^{M,\overline{t}(M)}(s).$$

By Lemma 5.5 again this turns out to be a nonincreasing function on $[\overline{t}(M), T]$. For $s \in [\underline{t}(M), \overline{t}(M)]$ we have

$$\begin{aligned} \partial_s \phi^{C^M}(s) &= (r+\beta)\phi^{C^M}(s) - e^{(r+\beta)s}\partial_y u(s, I^M(s))\beta e^{-\beta s} \\ &= (r+\beta)[\phi^{C^M}(s) - M] \\ &= (r+\beta)[\phi^{C^M}(s) - \phi^{C^M}(\overline{t}(M))] \end{aligned}$$

[3] In the singular case $\underline{t}(M) = \overline{t}(M) > 0$ we set $C^M \triangleq 0$, hence we have $\operatorname{supp} dC^M = \emptyset$ *instead* of $\operatorname{supp} dC^M = \{\underline{t}(M)\}$. This is consistent with the above formula for Y^{C^M}.

i.e. ϕ^{C^M} solves the ordinary differential equation $\partial_t f(t) = (r+\beta)[f(t) - f(\bar{t}(M))]$ which obviously admits only constant solutions. Thus $\phi^{C^M}(s) \equiv M$ on $[\underline{t}(M), \bar{t}(M)]$. By our optimality criterion (3) this already proves the case $\underline{t}(M) = 0 < \bar{t}(M)$ and yields that for $0 < \underline{t}(M) < \bar{t}(M)$ it suffices to show $\phi^{C^M}(s) \leq M$ on $[0, \underline{t}(M))$. Indeed the above considerations and the definition of $\underline{t}(M)$ give us

$$\begin{aligned} \phi^{C^M}(s) &= e^{(r+\beta)s}\left(\int_s^{\underline{t}(M)} \partial_y u(v, ye^{-\beta v})\beta e^{-\beta v}\,dv + e^{-(r+\beta)\underline{t}(M)}M\right) \\ &\leq e^{(r+\beta)s}\left(\int_s^{\underline{t}(M)} M(r+\beta)e^{-(r+\beta)v}\,dv + e^{-(r+\beta)\underline{t}(M)}M\right) \\ &= M \end{aligned}$$

for $s \in [0, \underline{t}(M))$.

3. Continuity of $M \mapsto \Psi(C^M)$ can be proved showing the continuity of $M \mapsto C^M \in \mathcal{C}$ when $\mathcal{C}$ is equipped with the topology of weak convergence of measures. For this we note that $Y^{C^M}(t)$ $(0 \leq t \leq T)$ can be written in the more compact form

$$Y^{C^M}(t) = ye^{-\beta t} \vee I^M(t \wedge \bar{t}(M))e^{-\beta(t-\bar{t}(M))^+}.$$

Obviously, this is continuous in $M > 0$. Now the formula

$$C^M(t) = \frac{1}{\beta}(Y^{C^M}(t) - y) + \int_0^t Y^{C^M}(s)\,ds \quad (0 \leq t \leq T)$$

yields continuity of $C^M(t)$ in $M > 0$ for each $t \in [0, T]$ which is sufficient for continuity of $M \mapsto C^M$ w.r.t. the topology of weak convergence of measures.

4. It remains to show the surjectivity of $M \mapsto \Psi(C^M)$. Since this function is continuous this follows easily by $\Psi(C^M) = 0$ for M sufficiently large and $\Psi(C^M) \geq I^M(0) - y \uparrow \infty$ $(M \downarrow 0)$.

□

Corollary 5.9 *Under Assumption 5.1 the solution to the investor's optimization problem* (2) *with $\psi(t) = e^{-rt}$ for some constant $r \geq 0$ is given by the uniquely determined C^M satisfying $\Psi(C^M) = w$.*

□

6 Complete Solution for Power Utilities

Let us now illustrate the above method by the case of a separable power felicity function, i.e.

$$u(t, y) \stackrel{\Delta}{=} e^{-\delta t}\frac{1}{\alpha}y^\alpha \quad (0 \leq t \leq T, y > 0)$$

for some constants $\delta > 0$, $\alpha \in (0,1)$. Simple calculations yield

$$\begin{aligned} \partial_y u(t,y) &= e^{-\delta t} y^{-(1-\alpha)}, \quad i(t,z) = (e^{\delta t} z)^{-\frac{1}{1-\alpha}} \\ \partial_y^2 u(t,y) &= -(1-\alpha) e^{-\delta t} y^{-(2-\alpha)}, \quad \partial_y \partial_t u(t,y) = -\delta e^{-\delta t} y^{-(1-\alpha)} \\ \mathcal{L} u(t,y) &= (r + \beta(1-\alpha) - \delta) e^{-\delta t} y^{-(1-\alpha)} \end{aligned}$$

and

$$\begin{aligned} I^M(t) &= \left(M \tfrac{r+\beta}{\beta} e^{(\delta - r)t}\right)^{-\frac{1}{1-\alpha}}, \\ \phi^{M,t}(s) &= M \tfrac{r+\beta}{\delta+\alpha\beta} e^{-(r+\beta(1-\alpha)-\delta)t} \left(e^{(r+\beta(1-\alpha)-\delta)s} - e^{-(\delta+\alpha\beta)T} e^{(r+\beta)s}\right), \\ \Phi^M(t) &= M \tfrac{r+\beta}{\delta+\alpha\beta} \left(1 - e^{-(\delta+\alpha\beta)(T-t)}\right). \end{aligned} \tag{13}$$

Thus an investor with such a felicity function u has preferences satisfying our standing Assumption 5.1 iff $r + \beta(1-\alpha) - \delta > 0$, a condition also appearing in HHK, p. 426.

Remark 6.1 *Although our method does not apply to the case $r + \beta(1-\alpha) - \delta \le 0$ we nevertheless can show that in this case it is optimal to consume the whole wealth in one single gulp at time $t = 0$. Indeed, this follows by our general optimality criterion* (3) *because for $C \equiv w \in \mathcal{A}(w)$ we have $\phi^C = \phi^{M,0}$ for some suitable $M > 0$ and $\phi^{M,0}(s)$ decreases in s if $r + \beta(1-\alpha) - \delta \le 0$, confer* (13).

So let us assume that $r + \beta(1-\alpha) - \delta > 0$ and apply our method. First we determine the last time of consumption $\bar{t}(M)$ for each $M > 0$. It is easy to see that $\Phi^M(t) = M$ has a solution $t \in (0, T]$ iff $\bar{\tau} > 0$ where

$$\bar{\tau} \triangleq T - \frac{1}{\delta + \alpha\beta} \log \frac{r+\beta}{r + \beta(1-\alpha) - \delta} < T \tag{14}$$

and in this case it coincides with this quantity. Hence, we have $\bar{t}(M) = \bar{\tau}^+$ for all $M > 0$. This tells us in particular that, surprisingly, the last time of consumption does not depend on the investor's initial wealth w — a fact which seems to rely heavily on the special structure (separability, homogenity) of the above choice of u.

For $\bar{t}(M) = 0$, i.e. in case $\bar{\tau} \le 0$, we know it is optimal to consume the whole wealth immediately at time $t = 0$.

Thus it remains to treat the case $\bar{t}(M) \equiv \bar{\tau} > 0$. By (12) the first time of consumption $\underline{t}(.)$ can easily be seen to be given by $\underline{t}(M) = \underline{\tau}(M)^+ \wedge \bar{\tau}^+$ $(M > 0)$ where

$$\underline{\tau}(M) \triangleq \frac{1}{r + \beta(1-\alpha) - \delta} \log\left(M \frac{r+\beta}{\beta} y^{1-\alpha}\right) \quad (M > 0). \tag{15}$$

We have

$$\underline{\tau}(M) > 0 \text{ iff } M > \underline{M} \triangleq \frac{\beta}{r+\beta} y^{-(1-\alpha)}$$

and

$$\underline{\tau}(M) < \overline{\tau} \text{ iff } M < \overline{M} \triangleq \frac{\beta}{r+\beta} y^{-(1-\alpha)} e^{(r+\beta(1-\alpha)-\delta)\overline{\tau}}$$

which leads us to distinguish the following three cases to determine C^M $(M > 0)$:

(i) For $0 < M \leq \underline{M}$, i.e. $\underline{t}(M) = 0 < \overline{t}(M)$, C^M is given by supp $dC^M = [0, \overline{\tau}]$,

$$\Delta C^M(0) = \frac{1}{\beta}\left((M\frac{r+\beta}{\beta})^{-\frac{1}{1-\alpha}} - y\right)$$

and, since $dC^M(t) = -\frac{1}{\beta}\frac{\mathcal{L}u}{\partial_y^2 u}(t, I^M(t))\,dt$ on the interior of supp dC^M by Remark 5.4,

$$dC^M(t) = \left(1 - \frac{\delta - r}{\beta(1-\alpha)}\right)\left(M\frac{r+\beta}{\beta}e^{(\delta-r)t}\right)^{-\frac{1}{1-\alpha}} dt \quad (t \in (0, \overline{\tau})).$$

Consequently,

$$\Psi(C^M) = \left(M\frac{r+\beta}{\beta}\right)^{-\frac{1}{1-\alpha}}\left\{\frac{1}{\beta} + \frac{r+\beta(1-\alpha)-\delta}{\beta(\delta-\alpha r)}\left(1 - e^{-\frac{\delta-\alpha r}{1-\alpha}\overline{\tau}}\right)\right\} - \frac{y}{\beta}$$

if $\delta \neq \alpha r$, and

$$\Psi(C^M) = \left(M\frac{r+\beta}{\beta}\right)^{-\frac{1}{1-\alpha}}\left\{\frac{1}{\beta} + \left(1 - \frac{\delta - r}{\beta(1-\alpha)}\right)\overline{\tau}\right\} - \frac{y}{\beta}$$

in case $\delta = \alpha r$.

(ii) For $\underline{M} < M < \overline{M}$, i.e. $0 < \underline{t}(M) < \overline{t}(M)$, C^M has support $[\underline{\tau}(M), \overline{\tau}]$ where

$$dC^M(t) = \left(1 - \frac{\delta - r}{\beta(1-\alpha)}\right)\left(M\frac{r+\beta}{\beta}e^{(\delta-r)t}\right)^{-\frac{1}{1-\alpha}} dt.$$

Hence,

$$\begin{aligned}\Psi(C^M) \quad = \quad & \frac{r+\beta(1-\alpha)-\delta}{\beta(\delta-\alpha r)}\left\{\left(M\frac{r+\beta}{\beta}\right)^{-\frac{r+\beta}{r+\beta(1-\alpha)-\delta}} y^{-\frac{\delta-\alpha r}{r+\beta(1-\alpha)-\delta}}\right.\\ & \left. - \left(M\frac{r+\beta}{\beta}\right)^{-\frac{1}{1-\alpha}} e^{-\frac{\delta-\alpha r}{1-\alpha}\overline{\tau}}\right\}\end{aligned}$$

for $\delta \neq \alpha r$, and

$$\Psi(C^M) = \left(1 - \frac{\delta - r}{\beta(1-\alpha)}\right)\left(M\frac{r+\beta}{\beta}\right)^{-\frac{1}{1-\alpha}}(\overline{\tau} - \underline{\tau}(M))$$

if $\delta = \alpha r$.

(iii) Finally, for $M \geq \overline{M}$, i.e. $0 < \underline{t}(M) = \overline{t}(M)$, we have $C^M = 0$ and $\Psi(C^M) = 0$ as pointed out in Theorem 5.8.

Now, given the investor's initial capital $w \geq 0$ we only have to find $M > 0$ with $\Psi(C^M) = w$ — the corresponding C^M has to be the optimal consumption plan by Theorem 5.8. This can easily be done using the above explicit formulas for $\Psi(C^M)$ $(M > 0)$.

Let us summarize the above results in

Theorem 6.2 *Assume the investor's index of past consumption Y^C $(C \in \mathcal{C})$ is given by* (6) *and his felicity function is of the form $u(t, y) = e^{-\delta t}\frac{1}{\alpha}y^{\alpha}$ for some constants $\delta > 0$, $\alpha \in (0, 1)$. Then he optimally consumes his whole wealth $w > 0$ in one single gulp at time $t = 0$ iff*

$$r + \beta(1 - \alpha) \leq \delta$$

or

$$\overline{\tau} \text{ given by (14) is not strictly positive.}$$

Otherwise, if

$$w \geq \hat{w} \triangleq \begin{cases} \frac{r+\beta(1-\alpha)-\delta}{\beta(\delta-\alpha r)}\left(1 - e^{-\frac{\delta-\alpha r}{1-\alpha}\overline{\tau}}\right) y & \text{for} \quad \delta \neq \alpha r \\ \left(1 - \frac{\delta-r}{\beta(1-\alpha)}\right)\overline{\tau} y & \qquad\quad \delta = \alpha r \end{cases}$$

it is optimal to have an initial consumption gulp of size

$$\Delta C(0) = \frac{w - \hat{w}}{(y + \beta w)(y + \beta \hat{w})} y$$

and to consume at rates

$$dC(t) = \left(1 - \frac{\delta - r}{\beta(1 - \alpha)}\right) \frac{y + \beta w}{y + \beta \hat{w}} y e^{-\frac{\delta-r}{1-\alpha}t}\, dt$$

until time $t = \overline{\tau}$.

In case $w < \hat{w}$ the investor optimally waits until time $\underline{\tau}(M^*) > 0$ defined by (15) *when he starts consuming at rates*

$$dC^{M^*}(t) = \left(1 - \frac{\delta - r}{\beta(1 - \alpha)}\right) \left(M^* \frac{r + \beta}{\beta} e^{(\delta-r)t}\right)^{-\frac{1}{1-\alpha}} dt$$

until time $t = \overline{\tau}$. Here $M^* > 0$ is determined as $M^* \triangleq \frac{\beta}{r+\beta}K$ where $K > 0$ is the (unique) solution to

$$K^{-\frac{r+\beta}{r+\beta(1-\alpha)-\delta}} y^{-\frac{\delta-\alpha r}{r+\beta(1-\alpha)-\delta}} - K^{-\frac{1}{1-\alpha}} e^{-\frac{\delta-\alpha r}{1-\alpha}\overline{\tau}} = \frac{\beta(\delta - \alpha r)}{r + \beta(1 - \alpha) - \delta} w$$

if $\delta \neq \alpha r$, and

$$K^{-\frac{1}{1-\alpha}} \left\{ \frac{1}{\beta} \log K + \frac{1 - \alpha}{\beta} \log y - (r + \beta(1 - \alpha) - \delta)\overline{\tau} \right\} = -(1 - \alpha) w$$

in case $\delta = \alpha r$.

Remark 6.3 *The most surprising feature of the above solution might be that in contrast to both, the infinite horizon setting in HHK and to a setup using time-additive utilities and habit-formation (as in, e.g., [1], [3]), in our context the investor optimally refrains from consumption at a certain point in time. The solution in HHK may be recovered from ours letting* $T \uparrow \infty$ *assuming (as in HHK) that* $\alpha r < \delta < r + \beta(1-\alpha)$*. To illustrate the difference to the time-additive setup, one might consider the case* $\delta = r$ *and initial standard of living* $y = 0$*. Then a time-additive utility maximizer consumes at constant rates* $c^{add} = \frac{rw}{1-e^{-rT}}$*. A HHK-utility maximizer takes an initial gulp of size* $\frac{rw}{\beta(r+\beta(1-e^{-r\overline{\tau}}))}$ *and consumes afterwards at constant rates* $c^{HHK} = \frac{rw}{1-e^{-r\overline{\tau}}}$ *which are higher than* c^{add} *because of* $\overline{\tau} < T$*. Thus, a HHK-utility maximizer transfers wealth from the distant future to the present. The reason for this is, that he still obtains utility from past consumption even after stopping consumption. Loosely speaking, being old, he enjoys having had a good time as a young man.*

References

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